Correlation Risk and Optimal Portfolio Choice

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ABSTRACT

We develop a new framework for multivariate intertemporal portfolio choice that allows us to derive optimal portfolio implications for economies in which the degree of correlation across industries, countries, or asset classes is stochastic. Optimal portfolios include distinct hedging components against both stochastic volatility and correlation risk. We find that the hedging demand is typically larger than in univariate models, and it includes an economically significant covariance hedging component, which tends to increase with the persistence of variance–covariance shocks, the strength of leverage effects, the dimension of the investment opportunity set, and the presence of portfolio constraints.

This paper develops a new multivariate modeling framework for intertemporal portfolio choice under a stochastic variance–covariance matrix. We consider an incomplete market economy, in which stochastic volatilities and stochastic correlations follow a multivariate diffusion process. In this setting, volatilities and correlations are conditionally correlated with returns, and optimal portfolio strategies include distinct hedging components against volatility and correlation risk. We solve the optimal portfolio problem and provide simple closed-form solutions that allow us to study the volatility and covariance hedging demands in realistic asset allocation settings. We document the importance of modeling the multivariate nature of second moments, especially in the context of optimal asset allocation, and find that the optimal hedging demand can be significantly different from the one implied by more common models with constant correlations or single-factor stochastic volatility.

An important thread within the asset pricing literature has documented the characteristics of the time variation in the covariance matrix of asset

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returns. The importance of solving portfolio choice models taking into account the time variation in volatilities and correlations is highlighted by Ball and Torous (2000), who study empirically the comovement of a number of international stock markets. They find that the estimated correlation structure changes over time depending on economic policies, the level of capital market integration, and relative business cycle conditions. They conclude that ignoring the stochastic component of the correlation can easily imply erroneous portfolio choice and risk management decisions.

A revealing example of the importance of modeling time-varying correlations is offered by the comovement of financial markets during the recent 2007 to 2008 financial markets crisis. During the period between April 2005 and April 2008, the sample average correlation of S&P500 and Nikkei (FTSE) weekly stock market returns has been less than 0.20. However, its time variation has been large: Since summer 2007 international equity correlations increased dramatically, with correlations between the S&P500 and the FTSE reaching a value close to 0.80 for the quarter ending in April 2008 (see Panel 1 of Figure 1).

Another feature highlighted by the data is that correlation processes seem far from being independent: As the correlation with the FTSE has increased, the correlation with the Nikkei has also increased, reaching its highest value of 0.60 in the same month. A last important feature, highlighted in Panel 2 of Figure 1, is the correlation leverage effect: Correlations of stock returns tend to be higher in phases of market downturn. Although some of these empirical facts have been documented in the literature (see, e.g., Harvey and Siddique 2000, Roll 1988, and Ang and Chen 2002), little is known about (a) the solution of the optimal portfolio choice problem when correlations are stochastic and (b) the extent to which stochastic correlations affect the characteristics of optimal portfolios in realistic economic settings.

An extensive literature has explored the implications of stochastic volatility for intertemporal portfolio choice. However, the implications of stochastic correlations in a multivariate setting are still not well known. In part, this is due to the difficulty in formulating a flexible and tractable model satisfying the tight nonlinear constraints implied by a well-defined correlation process. We use our model to address a number of questions on the role of correlation hedging for intertemporal portfolio choice:

1Longin and Solnik (1995) reject the null hypothesis of constant international stock market correlations and find that they increase in periods of high volatility. Ledoit, Santa-Clara, and Wolf (2003) show that the level of correlation depends on the phase of the business cycle. Erb, Harvey, and Viskanta (1994) find that international markets tend to be more correlated when countries are simultaneously in a recessionary state. Moskowitz (2003) documents that covariances across portfolio returns are highly correlated with NBER recessions and that average correlations are highly time varying. Ang and Chen (2002) show that the correlation between U.S. stocks and the aggregate U.S. market is much higher during extreme downside movements than during upside movements. Barndorff-Nielsen and Shephard (2004) find similar results. Bekaert and Harvey (1995, 2000) provide direct evidence that market integration and financial liberalization change the correlation of emerging markets’ stock returns with the global stock market index.
First, what is the economic importance of stochastic variance–covariance risk for intertemporal portfolio choice? We estimate the model using a data set of international stock and U.S. bond returns and find that—even for a moderate number of assets—the hedging demand can be about four to five times...
larger than in univariate stochastic volatility models. This has two reasons. First, covariance hedging can count for a substantial part of the total hedging demand. Its importance tends to increase with the strength of leverage effects and the dimension of the investment opportunity set. Second, our findings show not only that the joint features of volatility and correlation dynamics are better described by a multivariate model with nonlinear dependence and leverage, but also that they play an important role in the implied optimal portfolios. For instance, in a univariate stochastic volatility model we find that the estimated total hedging demand for S&P500 futures of investors with a relative risk aversion of eight and an investment horizon of 10 years is only about 4.8% of the myopic portfolio. This finding is consistent with the results in Chacko and Viceira (2005). However, in a multivariate (three risky assets) model, the total hedging demand for S&P500 futures is 28% and the covariance hedging demand is 16.9% of the myopic portfolio.

Second, how do both optimal investment in risky assets and covariance hedging demand vary with respect to the investment horizon? This question is important for optimal life-cycle decisions as well as for pension fund managers. We find that the absolute correlation hedging demand increases with the investment horizon. If the correlation hedging demand is positive (negative), this feature implies an optimal investment in risky assets that increases (decreases) in the investment horizon. For instance, in a multivariate model with three risky assets, the estimated total hedging (covariance hedging) demand for S&P500 futures of investors with a relative risk aversion of eight is only about 6.3% (4.5%) of the myopic portfolio at horizons of 3 months. For horizons of 10 years the total hedging demand increases to 28%.

Third, what is the link between the persistence of correlation shocks and the demand for correlation hedging? The persistence of correlation shocks varies across markets. In highly liquid markets like the Treasury and foreign exchange markets, which are less affected by private information issues, correlation shocks are less persistent. In other markets, frictions such as asymmetric information and differences in beliefs about future cash flows make price deviations from the equilibrium more difficult to be arbitraged away. Examples include both developed and emerging equity markets. Consistent with this intuition, we find that the optimal hedging demand against covariance risk increases with the degree of persistence of correlation shocks.

Fourth, what is the impact of discrete trading and portfolio constraints on correlation hedging demands? In the absence of derivative instruments to complete the market, we find that the covariance hedging demand in continuous-time and discrete-time settings are comparable. Simple short-selling constraints tend to reduce the covariance hedging demand of risk-tolerant investors, typically by a moderate amount, but Value-at-Risk (VaR) constraints can even reinforce the covariance hedging motive. For instance, in the unconstrained discrete-time model with two risky assets the estimated total hedging (covariance hedging) demand for S&P500 futures of investors with relative risk aversion of two is about 12.5% (4.6%) of the myopic portfolio at horizons of
2 years. In the VaR-constrained setting, the total hedging (covariance hedging) demand increases to 16.7% (8.1%).

This paper draws upon a large literature on optimal portfolio choice under a stochastic investment opportunity set. One set of papers studies optimal portfolio and consumption problems with a single risky asset and a riskless deposit account. Portfolio selection problems with multiple risky assets have been considered in a further series of papers, but the majority of these are based on the assumption that volatility and correlation are constant. Examples include Brennan and Xia (2002), who study optimal asset allocation under inflation risk, and Sangvinatsos and Watcher (2005), who investigate the portfolio problem of a long-run investor with both nominal bonds and a stock. A notable exception to the constant volatility assumption is Liu (2007), who shows that the portfolio problem can be characterized by a sequence of differential equations in a model with quadratic returns. However, to solve in closed form a concrete model with a riskless asset, a risky bond, and a stock, he assumes independence between the state variable driving interest rate risk and the additional risk factor influencing the volatility of the stock return. Under these assumptions, correlations are restricted to being functions of stock and bond return volatilities. Therefore, optimal hedging portfolios do not allow volatility and correlation risk to have separate roles. We investigate an economy with multivariate risk factors, where correlations are nonredundant sources of risk and several risky assets can be conveniently considered.

We follow a new approach in modeling stochastic variance–covariance risk and directly specify the covariance matrix process as a Wishart diffusion process. This process can reproduce several of the empirical features of returns covariance matrices highlighted above. At the same time, it is sufficiently tractable to grant closed-form solutions to the optimal portfolio problem, which we can easily interpret economically. The Wishart process is a single-regime mean-reverting matrix diffusion, in which the strength of the mean reversion can generate different degrees of persistence in volatilities, correlations, and co-volatilities. A completely different approach to modeling co-movement in portfolio choice relies on either a Markov switching regime in correlations or

\[ \text{Kim and Omberg (1996) solve the portfolio problem of an investor optimizing utility of terminal wealth, where the riskless rate is constant and the risky asset has a mean reverting Sharpe ratio and constant volatility. Wachter (2002) extends this setting to allow for intermediate consumption and derives closed-form solutions in a complete markets setting. Chacko and Viceira (2005) relax the assumption on both preferences and volatility. They consider an infinite horizon economy with Epstein–Zin preferences, in which the volatility of the risky asset follows a mean reverting square-root process. Liu, Longstaff, and Pan (2003) model events affecting market prices and volatility, using the double-jump framework in Duffie, Pan, and Singleton (2000). They show that the optimal policy is similar to that of an investor facing short selling and borrowing constraints, even if none are imposed. Although their approach allows for a rather general model with stochastic volatility, they focus on an economy with a single risky asset.} \]

\[ \text{See Bru (1991). The convenient properties of Wishart processes for modeling multivariate stochastic volatility in finance are exploited first by Gouriéroux and Sufana (2004); Gouriéroux, Jasiak, and Sufana (2009) provide a thorough analysis of the properties of Wishart processes, both in discrete and continuous time.} \]
the introduction of a sequence of unpredictable joint Poisson shocks in asset returns. Ang and Bekaert (2002) consider a dynamic portfolio model with two i.i.d. switching regimes, one of which is characterized by higher correlations and volatilities. Das and Uppal (2004) study systemic risk, modeled as an unpredictable common Poisson shock, in a setting with a constant opportunity set and in the context of international equity diversification.

The article proceeds as follows: Section I describes the model, the properties of the implied correlation process, and the solution to the portfolio problem. In Section II we estimate the model in a real data example and quantify the portfolio impact of correlation risk. Section III discusses extensions that study the impact of discrete rebalancing and investment constraints on correlation hedging. Section IV concludes. An Internet Appendix contains the proofs of all the Theorems and Propositions. This appendix, moreover, contains several additional results and extensions of the main model.4

I. The Model

An investor with constant relative risk aversion (CRRA) utility over terminal wealth trades three assets, a riskless asset with instantaneous riskless return \( r \) and two risky assets, in a continuous-time frictionless economy on a finite time horizon \([0, T]\).5 The dynamics of the price vector \( S = (S_1, S_2)' \) are described by the bivariate stochastic differential equation:

\[
\frac{dS(t)}{I_S} = \left[ \left( r \widehat{1}_2 + \Lambda(\Sigma, t) \right) dt + \Sigma^{1/2}(t) dW(t) \right]. \quad I_S = \text{Diag}[S_1, S_2].
\] (1)

where \( r \in \mathbb{R}_+ \), \( \widehat{1}_2 = (1, 1)' \), \( \Lambda(\Sigma, t) \) is a vector of possibly state-dependent risk premia, \( W \) is a standard two-dimensional Brownian motion, and \( \Sigma^{1/2} \) is the positive square root of the conditional covariance matrix \( \Sigma \). The investment opportunity set is stochastic because of the time-varying market price of risk \( \Sigma^{-1/2}(t) \Lambda(\Sigma, t) \). The constant interest rate assumption can easily be relaxed. Such an extension is investigated in Internet Appendix C. The diffusion process for \( \Sigma \) is detailed below. Let \( \pi(t) = (\pi_1(t), \pi_2(t))' \) denote the vector of shares of wealth \( X(t) \) invested in the first and the second risky asset, respectively. The agent's wealth evolves according to

\[
\frac{dX(t)}{X(t)} = \left[ r + \pi(t)' \Lambda(\Sigma, t) \right] dt + X(t)\pi(t)' \Sigma^{1/2}(t) dW(t).
\] (2)

The agent selects the portfolio process \( \pi \) that maximizes CRRA utility of terminal wealth, with RRA coefficient \( \gamma \). If \( X_0 = X(0) \) denotes the initial wealth and

4An Internet Appendix for this article is online in the “Supplements and Datasets” section at http://www.afajof.org/supplements.asp. It is organized in five subappendices, labeled with capital letters from A to E. The first section contains the proofs, the following sections the extensions. In the text, we refer to these subappendices as “Internet Appendix x”, where x is the letter that identifies the subappendix.

5Our analysis extends to opportunity sets consisting of any number of risky assets and correlations, without affecting the existence of closed-form solutions and their general structure. We consider a two-dimensional setting to keep our notation simple and focus on the key economic intuition and implications of the solution.
\[ \Sigma_0 = \Sigma(0) \] the initial covariance matrix, the investor’s optimization problem is

\[ J(X_0, \Sigma_0) = \sup_{\pi} \mathbb{E} \left[ \frac{X(T)^{1-\gamma} - 1}{1 - \gamma} \right], \quad (3) \]

subject to the dynamic budget constraint (2).

**A. The Stochastic Variance Covariance Process**

To model stochastic covariance matrices, we use the continuous-time Wishart diffusion process introduced by Bru (1991). This process is a matrix-valued extension of the univariate square-root process that gained popularity in the term structure and stochastic volatility literature; see, for instance, Cox, Ingersoll, and Ross (1985) and Heston (1993). Let \( B(t) \) be a \( 2 \times 2 \) standard Brownian motion. The diffusion process for \( \Sigma \) is defined as

\[ d\Sigma(t) = [\Omega \Omega' + M \Sigma(t) + \Sigma(t)M']dt + \Sigma^{1/2(t)}dB(t)Q + QdB(t)\Sigma^{1/2(t)}, \quad (4) \]

where \( \Omega, M, \) and \( Q \) are \( 2 \times 2 \) square matrices (with \( \Omega \) invertible). Matrix \( M \) drives the mean reversion of \( \Sigma \) and is assumed to be negative semidefinite to ensure stationarity. Matrix \( Q \) determines the co-volatility features of the stochastic variance–covariance matrix of returns.

Process (4) satisfies several properties that make it ideal to model stochastic correlation in finance. First, if \( \Omega \Omega' \gg QQ \), then \( \Sigma \) is a well-defined covariance matrix process. Second, if \( \Omega \Omega' = kQQ \) for some \( k > n - 1 \), then \( \Sigma(t) \) follows a Wishart distribution. Third, the process (4) is affine in the sense of Duffie and Kan (1996) and Duffie, Filipovic, and Schachermayer (2003). This feature implies closed-form expressions for all conditional Laplace transforms. Fourth, if \( d\ln S_t \) is a vector of returns with a Wishart covariance matrix \( \Sigma(t) \), then the variance of the return of a portfolio \( \pi \) is a Wishart process. Fifth, processes (1) and (4) can feature some important empirical regularities of financial asset returns documented in the literature, such as leverage and co-leverage.

To model leverage effects, we assume a nonzero correlation between innovations in stock returns and innovations in the variance–covariance process. Specifically, we define the standard Brownian motion \( W(t) \) in the return dynamics as

\[ W(t) = \sqrt{1 - \tilde{\rho}'\tilde{\rho}}Z(t) + B(t)\tilde{\rho}, \quad (5) \]

where \( Z \) is a two-dimensional standard Brownian motion independent of \( B \) and \( \tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2)' \) is a vector of correlation parameters \( \tilde{\rho}_i \in [-1, 1] \) such that \( \tilde{\rho}'\tilde{\rho} \leq 1 \). Parameters \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) parameterize leverage effects in volatilities and correlations of the multivariate return process (1). Because \( n \) risky assets are available for investment and the covariance matrix depends on \( n(n+1)/2 \) independent Brownian shocks, the market is incomplete when \( n \geq 2 \).
B. Specification of the Risk Premium

The investment opportunity set can be stochastic due to changes in expected returns or changes in conditional variances and covariances. It is well known that to obtain closed-form solutions one needs to impose restrictions on the functional form of the squared Sharpe ratio. Affine squared Sharpe ratios imply affine solutions if the underlying state process is affine. Thus, we consider risk premium specifications \( \Lambda(\Sigma, t) \) that imply an affine dependence of squared Sharpe ratios on the state process.

We consider a setting with a constant market price of variance–covariance risk, \( \Lambda(\Sigma, t) = \Sigma(t)\lambda \) for \( \lambda = (\lambda_1, \lambda_2)' \in \mathbb{R}^2 \). This assumption implies squared Sharpe ratios that increase with volatility, but that can increase or decrease in the correlation depending on the sign of the prices of risk.\(^6\) We solve the dynamic portfolio problem implied by this specification in Section I.D.2. In Internet Appendix C, we solve the portfolio problem also under the assumption of a constant risk premium, \( \Lambda(\Sigma, t) = \mu^e, \mu^e = (\mu^e_1, \mu^e_2)' \in \mathbb{R}^2 \), and an affine matrix diffusion (4) for the precision process \( \Sigma^{-1} \). In this setting, the investment opportunity set is stochastic exclusively due to the stochastic covariance matrix.\(^7\) However, the disadvantage is that state variables are defined by means of \( \Sigma^{-1} \), which makes the interpretation of model parameters (e.g., in terms of volatility and correlation leverage effects) less straightforward.

C. Correlation Process and Leverage

An application of Itô’s Lemma gives us the correlation dynamics implied by the Wishart diffusion (4).

**PROPOSITION 1:** Let \( \rho \) be the correlation diffusion process implied by the covariance matrix dynamics (4). The instantaneous drift and conditional variance of \( d\rho(t) \) are given by

\[
\mathbb{E}_t[d\rho(t)] = \left[ E_1(t)\rho(t)^2 + E_2(t)\rho(t) + E_3(t) \right] dt,
\]

\[
\mathbb{E}_t[d\rho(t)^2] = \left[ (1 - \rho^2(t)) \left( E_4(t) + E_5(t)\rho(t) \right) \right] dt,
\]

where the coefficients \( E_1, E_2, E_3, E_4, \) and \( E_5 \) depend exclusively on \( \Sigma_{11}, \Sigma_{22}, \) and the model parameters \( \Omega, M, \) and \( Q \). The explicit expression for the coefficients \( E_1, \ldots, E_n \) in the correlation dynamics is derived in Internet Appendix A.

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\(^6\) The assumption of a constant market price of variance–covariance risk implies a positive risk-return tradeoff and embeds naturally the univariate model studied, among others, in Heston (1993) and Liu (2001).

\(^7\) Buraschi, Trojani, and Vedolin (2009) discuss a structural multiple trees economy in which this feature arises in a setting with time-varying economic uncertainty.
Proof: See Internet Appendix A.

The correlation dynamics are not affine, because the correlation is a nonlinear function of variances and covariances. The nonlinear drift and volatility functions imply nonlinear mean reversion and persistence properties, depending on the model parameters. Moreover, the drift and volatility coefficients are functions of the volatility of asset returns, showing the intrinsic multivariate nature of the correlation in our model. This property is a clear distinction from approaches that model correlations with a scalar diffusion (see, for instance, Driessen, Maenhout, and Vilkov (2007)).

Black’s volatility “leverage” effect, that is, the negative dependence between returns and volatility, is an empirical feature of stock returns, which has important implications for optimal portfolio choice. Roll’s (1988) correlation “leverage” effect, that is, the negative dependence between returns and average correlation shocks, is also a well-established stylized fact; see, for example, Ang and Chen (2002). In our model, these effects depend on parameter \( \bar{\rho} \) and the matrix \( Q \). To see this explicitly, one can use the properties of the Wishart process to obtain

\[
corr_t \left( \frac{dS_1}{S_1}, d\Sigma_{11} \right) = \frac{q_{11} \bar{\rho}_1 + q_{21} \bar{\rho}_2}{\sqrt{q_{11}^2 + q_{21}^2}} \quad \text{and} \quad \corr_t \left( \frac{dS_1}{S_1}, d\rho \right) = \frac{(q_{11} \bar{\rho}_1 + q_{21} \bar{\rho}_2)(1 - \rho^2(t))}{\sqrt{(E_t[d\rho^2]/dt)\Sigma_{22}(t)}},
\]

where for any \( i, j = 1, 2 \), parameters \( q_{ij} \) denote the \( ij \)th element of matrix \( Q \) and the expression for \( E_t[d\rho^2] \) is given in equation (7). The expressions for the second asset are symmetric, with \( q_{12} \) replacing \( q_{11} \) and \( q_{22} \) replacing \( q_{21} \), both in the first and the second equality. The element \( \Sigma_{11} \) replaces \( \Sigma_{22} \) in the second equality. From these formulas, the parameter vector \( Q' \bar{\rho} \) controls the dependence between returns, volatility, and correlation shocks: Volatility and correlation leverage effects arise for all assets if both components of \( Q' \bar{\rho} \) are negative.

D. The Solution of the Investment Problem

The first challenge in solving investment problem (3) subject to the covariance matrix dynamics (4) is that markets are incomplete. If we consider a market with only primary risky securities, then there is no (nondegenerate) specification of the model that allows the number of available risky assets to match the dimensionality of the Brownian motions.\(^8\)

\(^8\)In order to hedge volatility and correlation risk, one may consider derivatives with a pay-off that depends on the variances of a portfolio of the primary assets, for instance variance swaps or options on a “market” index; see for example Leippold, Egloff, and Wu (2007) for a univariate dynamic portfolio choice problem with variance swaps. If these derivatives completely span the state space generated by variances and covariances, then they can be used to complete the market and solve in closed form the optimal portfolio choice problem. The extent to which volatility and correlation hedging demands in the basic securities will arise depends on the ability of these additional derivatives to span the variance–covariance state space. Because variance swaps are
D.1. Incomplete Market Solution Approach

To solve the portfolio problem, we follow He and Pearson (1991) and solve the following static problem:

\[
J(X_0, \Sigma_0) = \inf_{\nu} \sup_{\pi} \mathbb{E} \left[ \frac{X(T)^{1-\gamma} - 1}{1-\gamma} \right],
\]

s.t. \( \mathbb{E} \left[ \xi_\nu(T)X(T) \right] \leq x, \)

where \( \nu \) indexes the set of all equivalent martingale measures and \( \xi_\nu \) is in the set of associated state price densities. The optimality condition for the optimization over \( \pi \) in problem (9) is \( X(T) = \left( \psi \xi_\nu(T) \right)^{-1/\gamma} \), where \( \psi \) is the multiplier of the constraint (10), so that we can focus without loss of generality on the solution of the problem:9

\[
\hat{J}(0, \Sigma_0) = \inf_{\nu} \mathbb{E} [\xi_\nu(T)^{\gamma-1}/\gamma].
\]

To obtain the value function of this problem in closed form, we take advantage of the fact that the Wishart process (4) is an affine stochastic process.

D.2. Exponentially Affine Value Function and Optimal Portfolios

The solution of the dynamic portfolio problem is exponentially affine in \( \Sigma \), with coefficients obtained as solutions of a system of matrix Riccati differential equations.

**PROPOSITION 2:** Given the covariance matrix dynamics in (4), the value function of problem (3) takes the form

\[
J(X_0, \Sigma_0) = X_0^{1-\gamma} \hat{J}(0, \Sigma_0)^{\gamma} - 1, \]

where the function \( \hat{J}(t, \Sigma) \) is given by

\[
\hat{J}(t, \Sigma) = \exp \left( B(t, T) + tr (A(t, T) \Sigma) \right),
\]

with \( B(t, T) \) and the symmetric matrix-valued function \( A(t, T) \) solving the system of matrix Riccati differential equations

\[
0 = \frac{dB}{dt} + tr[A\Omega'] - \frac{\gamma - 1}{\gamma} r,
\]

available only in some specific markets, in many cases variance–covariance risk is not likely to be completely hedgeable, which makes the incomplete market case of primary interest.

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9Results in Schroder and Skiadas (2003) imply that if the original optimization problem has a solution, the value function of the static problem coincides with the value function of the original problem. Cvitanic and Karatzas (1992) show that the solution to the original problem exists under additional restrictions on the utility function, most importantly that the relative risk aversion does not exceed one. Cuoco (1997) proves a more general existence result, imposing minimal restrictions on the utility function.
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\[ 0 = \frac{dA}{dt} + \Gamma' A + A \Gamma + 2A \Lambda A + C, \]  

(14)

under the terminal conditions \( B(T, T) = 0 \) and \( A(T, T) = 0 \). Constant matrices \( \Gamma, \Lambda, \) and \( C \), as well as the closed-form solution of the system of matrix Riccati differential equations (13) to (14), are reported in Internet Appendix A.

**Proof:** See Internet Appendix A.

**Remark:** In the term structure literature, it is well known that modeling correlated stochastic factors is not straightforward. Duffie and Kan (1996) show that for a well-defined affine process to exist, parametric restrictions on the drift matrix of the factor dynamics have to be satisfied. These features restrict the correlation structures that many affine models can fit (see, for example, Duffee, 2002). In the Dai and Singleton (2000) classification for affine \( A_n(n) \) models, restrictions need to be imposed for the model to be solved in closed form. This issue is well acknowledged also in the portfolio choice literature. For instance, Liu (2007) addresses it by assuming a triangular factor structure in an affine portfolio problem with two risky assets. Using the Wishart specification (4), we obtain a simple affine solution for problem (3), which does not imply excessive restrictions on the dependence of variance–covariance factors.\(^{10}\) Q.E.D

An advantage of the exponentially affine solution \( \hat{J} \) in Proposition 2 is that it allows for a simple description of the partial derivatives of the marginal indirect utility of wealth with respect to the variance–covariance factors. This property implies a simple and easily interpretable solution to the multivariate portfolio choice problem.

**PROPOSITION 3:** Let \( \pi \) be the optimal portfolio obtained under the assumptions of Proposition 2. It then follows that

\[ \pi = \frac{\lambda}{\gamma} + 2 \left[ (q_{11} \hat{\rho}_1 + q_{21} \hat{\rho}_2) A_{11} + (q_{12} \hat{\rho}_1 + q_{22} \hat{\rho}_2) A_{12} \right], \]  

(15)

where \( A_{ij} \) denotes the \( ij \)th component of the matrix \( A \), which characterizes the function \( \hat{J}(t, \Sigma) \) in Proposition 2, and the coefficients \( q_{ij} \) are the entries of the matrix \( Q \) appearing in the Wishart dynamics (4).

**Proof:** See Internet Appendix A.

The portfolio policy \( \pi = (\pi_1, \pi_2)' \) is the sum of a myopic demand and a hedging demand. The interpretation is simple and can easily be linked to Merton’s (1969) solution. The myopic portfolio is the optimal portfolio that would prevail in an economy with a constant opportunity set, that is, a constant covariance matrix. When the opportunity set is stochastic, the optimal portfolio also consists of an intertemporal hedging demand. This portfolio component reduces

\(^{10}\)The Wishart state space is useful also more generally, for example, for term structure modeling. Buraschi, Cieslak, and Trojani (2007) develop a completely affine model with a Wishart state space to explain several empirical regularities of the term structure at the same time.
the impact of shocks to the indirect utility of wealth. The size of intertemporal hedging depends on two components: (a) the extent to which investors’ marginal utility of wealth is indeed affected by shocks in the state variables, and (b) the extent to which these state variables are correlated with returns. Using Merton’s notation, the optimal hedging demand, denoted \( \pi_h \), can be written as

\[
\pi_h = - \sum_{i,j} \frac{J_{XX}}{XJ_{XX}} \Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{ij}) dt .
\]  

(16)

The term \(- \frac{J_{XX}}{XJ_{XX}} = - \frac{dx}{X} \frac{J_{XX}}{ds} = A_{ij}\) is a risk tolerance weighted sensitivity of the log of the indirect marginal utility of wealth with respect to the state variable \( \Sigma_{ij} \). The regression coefficient \( \Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{ij}) \) captures the ability of asset returns to hedge unexpected changes in this state variable. The hedging portfolio is zero if and only if either \( \frac{J_{XX}}{XJ_{XX}} = 0 \) for all \( i \) and \( j \) (e.g., log utility investors) or \( \Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{ij}) = 0 \). Using the properties of the Wishart process, the hedging portfolio then follows, in explicit form:

\[
\pi_h = \Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, \sum_{i,j} A_{ij} d\Sigma_{ij}) dt = 2 \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2 \\ q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2 \end{pmatrix} .
\]  

(17)

This is the second term in the sum on the right-hand side of formula (15). Hedging demands are generated by the willingness to hedge unexpected changes in the portfolio total utility due to shocks in the state variables \( \Sigma_{ij} \). Hedging demands proportional to \( A_{ij} \) are demands against unexpected changes in \( \Sigma_{ij} \). It follows that hedging portfolios proportional to \( A_{11} \) and \( A_{22} \) are volatility hedging portfolios, and hedging portfolios proportional to \( A_{12} \) are covariance hedging portfolios. The role of parameters \( Q \) and \( \tilde{\rho} \) in the hedging portfolio is clarified by writing equation (17) in the equivalent form:

\[
\pi_h = 2A_{11} \begin{pmatrix} q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2 \\ 0 \end{pmatrix} + 2A_{22} \begin{pmatrix} 0 \\ q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2 \end{pmatrix} + 2A_{12} \begin{pmatrix} q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2 \\ q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2 \end{pmatrix} .
\]  

(18)

The parameters \( Q \) and \( \tilde{\rho} \) determine the ability of asset returns to span shocks in risk factors, because they determine the regression coefficients \( \Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{ij}) \) in equation (16):

\[
\Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{11}) dt = 2 \begin{pmatrix} q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2 \\ 0 \end{pmatrix} ,
\]  

(19)

\[
\Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{22}) dt = 2 \begin{pmatrix} 0 \\ q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2 \end{pmatrix} ,
\]  

(20)

\[
\Sigma^{-1} \text{Cov}_t(I^{-1}_S dS, d\Sigma_{12}) dt = 2 \begin{pmatrix} q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2 \\ q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2 \end{pmatrix} .
\]  

(21)
By comparing (19) to (21) with (8), the sign of each component of $\Sigma^{-1} \text{Cov}(I_S^{-1}dS, d\Sigma_{ij})$ equals the sign of the co-movement between returns, variances, and correlations. It follows that the first and second columns of $Q$ impact the volatility and covariance hedging demand for the first and second assets, respectively, via the coefficient vectors $(q_{11}, q_{21})$ and $(q_{12}, q_{22})$. In contrast, parameter $\hat{\rho}$ directly impacts all hedging portfolios. Risky assets are better at spanning the risk of variance–covariance shocks when $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$ and $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$ are large in absolute value. Moreover, asset $i$ is a better hedging instrument against its stochastic volatility $\Sigma_{ii}$, and less so against shocks in the covariance $\Sigma_{ij}$, when the coefficient $q_{1i}\bar{\rho}_1 + q_{2i}\bar{\rho}_2$ is the largest one. Despite the simple hedging portfolio (15), a variety of other hedging implications can arise. For instance, when $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$ and $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$ are both negative, volatility and correlation leverage effects arise for all returns. However, if parameters $q_{11}\bar{\rho}_1 + q_{21}\bar{\rho}_2$ and $q_{12}\bar{\rho}_1 + q_{22}\bar{\rho}_2$ have mixed signs some returns will feature leverage effects, but others will not.

**D.3. Sensitivity of the Marginal Utility of Wealth to the State Variables**

The second determinant of the hedging demand is the sensitivity of the marginal utility of wealth to the state variables $\Sigma_{ij}$. This effect is summarized by the components $A_{ij}$. Therefore, it is useful to gain intuition on the dependence of $A_{ij}$ on the structural model parameters. For brevity, we focus on investors with risk aversion above one and a vector $\lambda$ such that $\lambda_1\lambda_2 \geq 0$. This setting includes the choice of parameters implied by the model estimation results in Section II.

**PROPOSITION 4:** Consider an investor with risk aversion parameter $\gamma > 1$. (i) The following inequalities, describing the properties of the sensitivity of the indirect marginal utility of wealth with respect to changes in the state variables $\Sigma_{ij}$, hold true: $A_{11}, A_{22} \leq 0$ and $|A_{12}| \leq |A_{11} + A_{22}|/2$. (ii) If it is additionally assumed that either $\lambda_1 \geq \lambda_2 \geq 0$ or $\lambda_1 \leq \lambda_2 \leq 0$, then $A_{12} \leq 0$ and $|A_{22}| \leq |A_{12}| \leq |A_{11}|$.

**Proof:** See Internet Appendix A.

This result describes the link between the indirect marginal utility of wealth and the state variables $\Sigma_{ij}$: $A_{ij} = -\frac{\partial J_{\Sigma_{ij}}}{\partial \Sigma_{ij}}$. This sensitivity is increasing with the sensitivity of the stochastic opportunity set, that is, the squared Sharpe ratio, to unexpected changes in $\Sigma_{ij}$. The squared Sharpe ratio is given by $\lambda_1^2 \Sigma_{11} + \lambda_2^2 \Sigma_{22} + 2\lambda_1\lambda_2 \Sigma_{12}$. Its sensitivity to the variance risk factor $\Sigma_{11}$ is highest when $|\lambda_1| \geq |\lambda_2|$, and vice versa. The sensitivity to the covariance risk factor is bounded by the absolute average sensitivity to the variance factors, because squared Sharpe ratios depend on $\Sigma_{12}$ via a loading that is twice the product of $\lambda_1$ and $\lambda_2$. To understand the sign of $A_{ij}$, recall that investors with risk aversion above one have a negative utility function bounded from above. Wealth homogeneity of the solution implies $J_X(t) = X(t)^{\gamma-1} \tilde{J}(t, \Sigma(t))^{1-\gamma}$, so that $J_{\Sigma_{ij}}$ and $J_{X\Sigma_{ij}}$ have the same sign. An increase in the variance $\Sigma_{ii}$ of one risky asset increases the squared Sharpe ratio of the optimal portfolio, but at the
same time it increases the squared Sharpe ratio variance. Investors with risk aversion above one dislike this effect, because ex ante they profit less from higher future Sharpe ratios than they suffer from higher future Sharpe ratio variances. These features imply the negative sign of $A_{ii}$. The sign of $J_{\Sigma_{12}}$ depends on how squared Sharpe ratios depend on $\Sigma_{12}$. If $\lambda_1\lambda_2 \geq 0$, $\Sigma_{12}$ affects the squared Sharpe ratios positively, which implies $A_{12} \leq 0$.

II. Hedging Stochastic Variance–Covariance Risk

We quantify volatility and covariance hedging for a realistic stock-bond portfolio problem in which a portfolio manager allocates wealth between the S&P500 Index futures contract, traded at the Chicago Mercantile Exchange, the Treasury bond futures contract, traded at the Chicago Board of Trade, and a riskless asset.

A. Data and Estimation Results

The model is estimated by GMM (Generalized Method of Moments) using the conditional moment conditions of the process, derived in closed form. The methodology is easily implemented and provides asymptotic tests for overidentifying restrictions. As a first step, we use the methodology proposed by Andersen et al. (2003) to obtain model-free realized volatilities and covariances from daily quadratic variations and covariances of log prices. The high-frequency data set we use is from “Price-Data” and “Tick-Data” and it includes tick-by-tick futures returns for the S&P500 Index and the 30-year Treasury bond, from January 1990 to October 2003.

We use both weekly and monthly returns, realized volatilities, and covariances, to investigate the impact of different exact discretizations of the model on the estimated parameters. Let $\theta := (\text{vec}(M)', \text{vec}(Q)', \lambda', \bar{\rho}')$. A GMM estimator of $\theta$ is given by

$$\hat{\theta} = \arg \min_{\theta} (\mu(\theta) - \mu_T)' V(\theta)(\mu(\theta) - \mu_T),$$

where $\mu_T$ is the vector of empirical moments implied by the historical returns and their realized variance–covariance matrices, and $\mu(\theta)$ is the theoretical vector of moments in the model. The term $V(\theta)$ is the GMM optimal weighting.

11This variance equals $4\lambda' \Sigma_{12} \lambda Q \bar{\rho}$, using the properties of Wishart processes.
12See Internet Appendix B. A closed form expression for the moments of the Wishart process can be found, for instance, in the Appendix of Buraschi et al. (2007).
14See the web pages www.grainmarketresearch.com and www.tickdata.com for details. When we estimate the model with three risky assets, in Internet Appendix E, we also use returns for the Nikkei225 Index futures contract.
Correlation Risk and Optimal Portfolio Choice

Table I
Estimation Results for the Model with Two Risky Assets

This table shows estimated matrices $M$ and $Q$ and vectors $\lambda$ and $\bar{\rho}$ for the returns dynamics (1) under the Wishart variance–covariance diffusion process

$$d\Sigma(t) = (\Omega' + M\Sigma(t) + \Sigma(t) M')dt + \Sigma^{1/2}(t)dB(t)Q + QdB(t)'\Sigma^{1/2}(t),$$

where we have set $\Omega' = kQ'Q$ and $k = 10$. Parameters are estimated by GMM using time series of returns and realized variance–covariance matrices for S&P500 Index and 30-year Treasury bond futures returns, computed at both a weekly and a monthly frequency. The detailed set of moment restrictions used for GMM estimation is given in Internet Appendix B. We report parameter estimates and their standard errors (in parentheses), together with the $p$-values for Hansen's $J$-test of overidentifying restrictions. An asterisk denotes estimated parameters that are not significant at the 5% significance level.

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<td>(0.0413)</td>
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<td>(0.028)</td>
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<td>Monthly: 0.38</td>
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matrix in the sense of Hansen (1982), estimated using a Newey–West estimator with 12 lags. We estimate $\theta$ using moment conditions that provide information about returns, their realized volatilities and correlations, and the leverage effects. The term $\mu_T$ consists of the following moment restrictions: unconditional risk premia of log-returns, unconditional first and second moments of variances and covariances of log-returns, and unconditional covariances between returns and each element of the variance–covariance matrix of returns. This leaves us with 17 moment restrictions for a 13-dimensional parameter vector, so that we have four overidentifying restrictions.

A.1. Basic Estimation Results

Table I presents results of our GMM model estimation. Hansen’s test of overidentifying restrictions does not reject the model specification at the weekly and monthly frequency.\footnote{We also estimate the model using daily returns, realized volatilities, and covariances. Results are reported in Internet Appendix E. In this case, the Hansen’s statistic rejected the model. We find that jumps in returns and realized conditional second moments are mainly responsible for this rejection, suggesting the misspecification of a pure multivariate diffusion in this context. The} The parameter estimates
support the multivariate specification of the correlation process. The null hypothesis that the volatility of volatility matrix $Q$ is identically zero is rejected at the 5% significance level, which supports the hypothesis of a stochastic correlation process. Parameter estimates for the components of $M$ are also almost all significant, supporting a multivariate mean reversion and persistence in variances and covariances. The estimated eigenvalues of matrix $M$ imply clear-cut evidence for two very different mean reversion frequencies, a high one and a low one, underlying the returns covariance matrix. All estimated eigenvalues are negative, which supports the stationarity of the variance–covariance process. The larger eigenvalues estimated with monthly returns imply a larger persistence of variance–covariance shocks at monthly frequencies. The estimated components of vector $\bar{\rho}$ are all negative and significant. Together with the positive point estimates for the coefficients of $Q$, this feature implies volatility and correlation leverage features for all risky assets. The point estimates for $Q$ highlight a large estimated parameter $q_{11}$. Thus, the first leverage parameter $\bar{q}_{11}\rho_1 + q_{12}\rho_2$ is about twice the size of the leverage parameter $\bar{q}_{12}\rho_1 + q_{22}\rho_2$. This implies that S&P500 returns are better vehicles to hedge their volatility risk than 30-year Treasury returns. At the same time, 30-year Treasury returns are better hedging instruments for hedging the covariance risk.

A.2. Estimated Correlation Process

Using the model parameter estimates, we can study the nonlinear dynamic properties of the implied correlation process. A convenient measure of the mean reversion properties of a nonlinear diffusion is given by its pull function—see Conley et al. (1997). The pull function $\varphi(x)$ of a process $X$ is the conditional probability that $X_t$ reaches the value $x + \epsilon$ before $x - \epsilon$, if initialized at $X_0 = x$. To first order in $\epsilon$, this probability is given by

$$\varphi(x) = \frac{1}{2} + \frac{\mu_X(x)}{2\sigma_X^2(x)} \epsilon + o(\epsilon), \quad (23)$$

where $\mu_X$ and $\sigma_X$ are the drift and the volatility function of $X$. Figure 2 presents nonparametric estimates of the pull function for the correlation and volatility processes of the S&P500 futures and 30-year Treasury futures returns, shifted by the factor 1/2 in equation (23).

16Evidence for a multifactor structure in the variance–covariance of asset returns is provided by Da and Schaumburg (2006), who apply Asymptotic Principal Component analysis to a panel of realized volatilities for U.S. stock returns. They find that three to four factors explain no more than 60% of the variation in realized volatilities and that the forecasting power of multifactor volatility models is superior to that of univariate ones. Similar findings are obtained by Andersen and Benzoni (2007). Calvet and Fisher (2007) develop an equilibrium model in which innovations in dividend volatility depend on shocks that decay with different frequencies. They show that this feature of volatility is crucial for the model’s forecasting performance.
Correlation Risk and Optimal Portfolio Choice

Figure 2. Pull function of the volatility and correlation processes. Panels 1, 4, and 7 show the nonparametric pull functions for weekly and monthly data frequencies (dotted and solid lines, respectively) based on the estimation procedure in Conley et al. (1997). Panels 1 and 4 present the pull function estimates for the conditional volatility of S&P500 Index and 30-year Treasury bond futures returns, respectively. Panel 7 plots the pull function estimate for their conditional correlation. Panels 2, 5, and 8 present the corresponding pull function estimates for volatilities and correlations, but based on a long time series of weekly volatilities and correlations simulated from the Wishart variance–covariance process (4) using the weekly parameter estimates in Table I. Panels 3, 6, and 9 present corresponding pull function estimates for volatilities and correlations, but based on a long time series of monthly volatilities and correlations simulated from the Wishart variance–covariance process (4) using the monthly parameter estimates in Table I. In Panels 2, 3, 5, 6, 8, and 9, the model implied pull functions are plotted together with a 95% confidence interval around the empirical pull functions of Panels 1, 4 and 7.
Each panel in the left column plots the estimated pull functions for volatilities and correlations for weekly and monthly data. The panels in the middle (right) column plot pull functions estimated from a long time series of observations simulated from our model. These pull functions are all inside a two-sided 95% confidence interval around the empirical pull functions, which indicates that our model can capture adequately the nonlinear mean reversion properties of volatilities and correlations in our data. Estimated pull functions for the correlation are highly asymmetric and are typically smaller in absolute value for positive correlations above 0.3 than for negative correlations below −0.4. This feature indicates a higher persistence of correlation shocks when correlations are positive and large. The pull function for the volatility of S&P500 futures returns in the first row of Figure 2 is almost flat and moderately positive for volatilities larger than 10%. On average, the pull function estimated for the volatility of 30-year Treasury futures returns tends to be larger in absolute value, which indicates a lower persistence of shocks in the volatility of Treasury futures returns.

Given the evidence of a nonlinear mean reversion of volatilities and correlations in the data, it is natural to ask whether univariate Markov continuous-time models can reproduce these features accurately. We estimate the Heston (1993) square-root processes for the volatility and an autonomous specification for the correlation process, as in Driessen et al. (2007). When we compute the model-implied pull functions, we find that (i) they are often outside the 95% confidence band around the empirical pull function and (ii) they are almost linear in shape, which is difficult to reconcile with our data. This suggests the importance of using an explicitly multivariate modeling approach.

B. The Size of Variance Covariance Hedging

We study the structure of the hedging demands based on our parameter estimates, and compute the optimal hedging components in Proposition 3 as a function of the relative risk aversion and the investment horizon.

B.1. Basic Results

Table II summarizes the estimated volatility and covariance hedging demands, as a percentage of the myopic portfolio allocations.

Overall, monthly estimates of the hedging demands are greater than weekly estimates. A more persistent variance–covariance process implies that variance–covariance shocks have more persistent effects on future squared Sharpe ratios and their volatility. This feature yields a higher absolute sensitivity of the marginal utility of wealth to variance–covariance shocks and greater absolute hedging demands on average.

Consider, for illustration purposes, the hedging demands estimated for monthly returns under an investment horizon of \( T = 5 \) years and a relative risk aversion parameter of six. The estimated risk premium for the S&P500

17Weekly estimates are reported in Internet Appendix E.
Table II
Optimal Hedging Demands for the Model with Two Risky Assets

This table shows optimal covariance and volatility hedging demands as a percentage of the myopic portfolio for different investment horizons and relative risk aversion parameters. The last column of each panel reports the myopic portfolio. We compute these demands for monthly parameter estimates reported in Table I. Each entry in the table is a vector with two components: the first component is the demand for the S&P500 Index futures and the second one is the demand for the 30-year Treasury futures.

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futures returns implies a higher loading of volatility than the one for the risk premium of 30-year Treasury futures returns: $\lambda_1 \geq \lambda_2$. Thus, the estimated sensitivity of the marginal utility of wealth to the returns volatility is highest for S&P500 futures returns: $|A_{11}| > |A_{22}|$. Moreover, according to the GMM parameter estimates, stocks are better instruments to hedge their volatility than bonds: $q_{11}\tilde{\rho}_1 + q_{21}\tilde{\rho}_2 \geq q_{12}\tilde{\rho}_1 + q_{22}\tilde{\rho}_2$. These features imply a higher volatility hedging demand for stocks (about 13% of the myopic portfolio) relative to the volatility hedging component for bonds (about 8% of the myopic portfolio). The total average volatility hedging demand is approximately 10.5% of the myopic portfolio, whereas the total average covariance hedging demand on the two
risky assets is slightly higher (about 11%). According to the GMM point estimates, bonds are better hedging vehicles than stocks to hedge covariance risk: 

\[ q_{12} \hat{\rho}_1 + q_{22} \hat{\rho}_2 \geq q_{11} \hat{\rho}_1 + q_{21} \hat{\rho}_2. \]

This effect determines the higher covariance hedging demand for bonds (about 17%) than for stocks (about 7%).

Given the evidence of a misspecification of univariate stochastic volatility models in our data, we compare the portfolio implications of our setting with those of univariate portfolio choice models with stochastic volatility; see Heston (1993) and Liu (2001), among others. This is an easy task because these models are nested in our setting for the special case in which the dimension of the investment opportunity set is equal to one. For each risky asset in our data set, we estimate these univariate stochastic volatility models by GMM.

Internet Appendix E presents the estimated volatility hedging demands as a percentage of the myopic portfolio. For illustration purposes, consider a relative risk aversion coefficient of \( \gamma = 8 \) and an investment horizon of \( T = 5 \) years. The volatility hedging demands estimated for the univariate models are 4.8% and 4%, respectively, for stocks and bonds. In the multivariate model, the corresponding pure volatility hedging demands are 13.6% and 8.8%, respectively, and the average total hedging demand is as large as 21.1%. One explanation for this finding is the very different mean reversion and persistence properties of second moments in the data, relative to those implied by univariate stochastic volatility models. A second reason is the fact that univariate models cannot capture the correlation and co-volatility dynamics, which generate a good portion of the total hedging demand. This is important because the optimal portfolio, as the results show, presents a substantial bias.

**B.2. Comparative Statics**

To get more insight into the determinants of hedging motives, it is useful to study comparative statics with respect to model parameters. To this end, we modify the value of these parameters in an interval of one sample standard deviation around the true parameter estimate, and compute the implied hedging demands.

It is natural to focus on parameters that have an impact on the indirect marginal utility sensitivities \( A_{ij} \), and the volatility and correlation leverage effects. Matrix \( M \) drives the persistence on the variance–covariance process, but leaves unaffected the leverage properties of asset returns. For brevity, we consider comparative statics with respect to the parameter \( m_{12} \). The matrix \( Q \) and vector \( \tilde{\rho} \) affect primarily the ability of each asset to span unexpected variance–covariance shocks. To study the effects of these parameters, we consider for brevity comparative statics with respect to \( q_{11} \) and both components of \( \tilde{\rho} \). In doing so, we decompose the total effect on hedging demands into one part due to a modification of the leverage properties of returns and one part due to the change in the marginal utility sensitivity coefficients \( A_{ij} \). The investment horizon we consider is \( T = 5 \) years and the relative risk aversion is \( \gamma = 6 \).

In the first row of Figure 3, the comparative statics with respect to \( m_{12} \) show that, ceteris paribus, covariance and volatility hedging demands increase with

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Figure 3. Comparative statics of optimal hedging demands. This figure shows comparative statics for optimal covariance hedging demands (Panels 1, 3, 5, and 6) and volatility hedging demands (Panels 2, 4, 7, and 8), obtained by letting the values of parameters $m_{12}$ (Panels 1 and 2), $q_{11}$ (Panel 3 and 4), $\bar{\rho}_1$, and $\bar{\rho}_2$ (Panels 7 to 10) vary in a confidence interval of one sample standard deviation around the monthly parameter estimates in Table I. $m_{ij}$ and $q_{ij}$ are the entries of parameter matrices $M$ and $Q$, respectively, appearing in the Wishart covariance matrix dynamics:

$$d\Sigma(t) = (\Omega^t + M\Sigma(t) + \Sigma(t) M')dt + \Sigma^{1/2}(t)dB(t)Q + Q dB(t)\Sigma^{1/2}(t).$$

$\bar{\rho}_1$ and $\bar{\rho}_2$ are the entries of the vector $\bar{\rho}$, which controls leverage by means of the following relation:

$$dW(t) = dB(t)\bar{\rho} + \sqrt{1 - \bar{\rho}'\bar{\rho}} dZ(t),$$

where $W(t)$ is the Brownian motion driving asset returns. In Panels 1 to 4, hedging demands for the S&P500 Index (30-year Treasury bond) futures are plotted with solid (dotted) lines. A relative risk aversion coefficient of five and an investment horizon of 6 years have been assumed.
For a value of \( m_{12} = 1.18 \), that is, one standard deviation above the GMM estimate, the covariance (volatility) hedging component increases to 8% (14%) for stocks and 23% (14%) for bonds.

This effect is due to the higher persistence of the variance–covariance process caused by an increase in \( m_{12} \), which implies a greater absolute marginal utility sensitivity to all variance–covariance risk factors.

The plots in the second row of Figure 3 present comparative statics with respect to \( q_{11} \). As \( q_{11} \) increases, the first risky asset becomes a better hedging instrument against its volatility, and the second risky asset becomes a better hedging instrument against covariance risk. Parameter \( q_{11} \) also has an effect on the marginal utility sensitivities \( A_{ij} \). We find that the higher variability of variance–covariance shocks implied by a higher parameter \( q_{11} \) lowers all absolute sensitivities \( |A_{ij}|, 1 \leq i, j \leq 2 \). However, this effect is considerably smaller than the one implied by the change in the leverage structure of asset returns. Consequently, as \( q_{11} \) increases we obtain a decreasing (increasing) covariance hedging demand for the S&P500 futures (30-year Treasury futures), but also an increasing (decreasing) volatility hedging component.

The comparative statics with respect to parameters \( \bar{\rho}_1 \) and \( \bar{\rho}_2 \) are given in the third and fourth rows of Figure 3. As \( \bar{\rho}_1 \) decreases, all assets become better hedging instruments against volatility and correlation risk. At the same time, the variance–covariance process under the minimax measure becomes more persistent, increasing each absolute sensitivity coefficient \( |A_{ij}| \). These two effects go in the same direction, but the effect on leverage is proportionally greater, and increases all volatility and covariance hedging demands. As \( \bar{\rho}_2 \) decreases, we observe almost no variation in volatility and covariance hedging demands. This follows from the fact that the leverage coefficients (18) and the minimax variance–covariance dynamics depend on \( \bar{\rho}_2 \) with a weight that is proportional to the parameters \( q_{12} \) and \( q_{22} \). According to our GMM estimates, these parameters are much smaller than \( q_{11} \) and \( q_{21} \).

**B.3. Time Horizon**

An important question addressed by the literature is how the optimal allocation in risky assets varies with respect to the investment horizon. For instance, Kim and Omberg (1996) show that for the investor with utility over terminal wealth, and for \( \gamma > 1 \), the optimal allocation increases with the investment horizon as long as the risk premium is positive. Wachter (2002) extends this result to the case of utility over intertemporal consumption.

When covariances are stochastic, it is reasonable that the optimal demand for hedging covariance risk could mitigate, or strengthen, the speculative components. Internet Appendix E reports estimated hedging demands for the S&P500 Index futures and the 30-year Treasury futures, as a function of the investment horizon. The total hedging demand of an investor with risk aversion \( \gamma > 1 \) increases with investment horizons of up to 5 years, where it approximately reaches a steady-state level. The reason for such a convergence is the stationarity of the Wishart process implied at our parameter estimates: Shocks in the
variance–covariance matrix do not seem to affect the transition density of the estimated variance–covariance process over horizons longer than 5 years. At very short horizons, for example, 3 months, hedging demands are small. For investment horizons of 5 years and higher, the total hedging demand is approximately 20% (25%) of the myopic portfolio for the S&P500 (Treasury) futures contracts. The covariance hedging demand for the 30-year Treasury futures increases quite quickly with the investment horizon, and it reaches a steady-state level of approximately 16% of the myopic demand. The covariance hedging demand for the S&P500 futures reaches a steady state of approximately 6.5% as the investment horizon increases.

B.4. Higher-Dimensional Portfolio Choice Settings

For simplicity we have so far focused on a portfolio choice setting with only two risky assets. However, it is important to gain some intuition on the relevance of volatility and covariance hedging when more risky assets are available for investment. The complexity of the portfolio setting increases as more volatility and covariance factors affect returns, which makes general statements and conclusions difficult. On the one hand, given that the number of covariance factors increases quadratically with the dimension of the investment universe, but the number of volatility factors increases only linearly, one could expect covariance hedging to become proportionally more important as the investment dimension rises. On the other hand, as the number of assets rises, one could also argue that covariance risk could become less important than volatility risk because the potential for portfolio diversification increases. The final effect depends on the extent to which shocks to the different covariance and volatility processes are diversifiable across assets.

We study quantitatively these issues in a concrete portfolio setting that includes also the Nikkei225 Index futures contract in the previous opportunity set, consisting of the S&P500 futures and the 30-year Treasury futures contracts. We estimate this three-dimensional version of model (1) to (4) using GMM. Estimated hedging demands for covariance and volatility hedging are given in Internet Appendix E.

For illustration purposes, consider a relative risk aversion of $\gamma = 6$ and an investment horizon of $T = 5$ years. The covariance hedging demand for the S&P500 futures is now approximately 12.5%, almost twice the hedging demand of 6.5% estimated in the two risky assets setting. The inclusion of the Nikkei225 futures sensibly lowers the covariance hedging demand for 30-year Treasury futures, which drops from 17% in the two-asset case to 6.1%. The covariance hedging demand for the Nikkei225 futures is approximately 13%. On average, these results imply a covariance hedging demand of about 11%. The intuition for these findings is simple: As the dimension of the investment opportunity set increases, the relative importance of covariance shocks to the squared Sharpe ratio of the optimal portfolio increases. The Nikkei225 provides a good opportunity to diversify domestic equity risk, under the assumption that covariances do not change. At the parameter estimates, an increased investment in
equity becomes increasingly coupled with a greater demand for hedging potential changes in these covariances. The volatility hedging demands for each asset are 9%, 6.1%, and 7.6%, respectively, and imply an average volatility hedging of about 7.5%. In the model with two risky assets, the average covariance hedging demand is about 11% and the average volatility hedging demand is about 10.5%.

Overall, these findings support the intuition that covariance hedging can become proportionally more important than volatility hedging as the dimension of the investment opportunity set rises.

III. Robustness and Extensions

In this section, we study the robustness of our findings, for example, with respect to a discrete-time solution of the portfolio choice problem or the inclusion of different portfolio constraints.

A. Risk Aversion

Our main findings do not depend on the choice of the relative risk aversion parameter used. Internet Appendix E contains a plot of hedging portfolios as a function of risk aversion. Hedging demands as a percentage of the myopic portfolio are monotonically increasing in the relative risk aversion coefficient, although the increase is small for relative risk aversion parameters above six. Average covariance hedging demands as a percentage of the myopic portfolio are typically higher than average volatility hedging demands. For instance, the average covariance (volatility) hedging demand for a relative risk aversion of 10 is approximately 12% (10.5%) of the myopic portfolio. Thus, although we assume a CRRA utility function to preserve closed-form optimal portfolios, our findings are likely to be even stronger in a setting with intertemporal consumption and Epstein–Zin recursive preferences because the risk aversion can be calibrated at a higher level without generating undesirable properties for the elasticity of intertemporal substitution.

B. Discrete-Time Solution and PortfolioConstraints

In our model, the optimal dynamic trading strategy is given by a portfolio that must be rebalanced continuously over time. In practice, this can at best be an approximation, because trading is only possible at discrete trading dates. Moreover, transaction costs, liquidity constraints, or policy disclosure considerations might further prevent investors from frequent portfolio rebalancing. Even if we do not model these frictions explicitly in our setting, it is instructive to analyze the impact of discrete trading on the optimal hedging strategy in the context of our model. We address this issue in Section I of Internet Appendix D. At a daily frequency, the hedging demands in the discrete-time model are virtually indistinguishable from the continuous-time hedging demands. The discrete-time optimal hedging demands for the monthly frequency
are close to the hedging demands computed from the continuous time model. These findings suggest that the main implications derived from the continuous time multivariate portfolio choice solutions are realistic even in the context of monthly rebalancing.

Portfolio constraints are useful to avoid unrealistic portfolio weights, which can potentially arise due to extreme assumptions on expected returns, volatilities, and correlations, or from inaccurate point estimates of the model parameters. Intuitively, constraints on short selling or the portfolio VaR tend to constrain the investor from selecting optimal portfolios that are excessively levered. Therefore, it is interesting to study such constraints and their impact on the volatility and covariance hedging demands in our setting. Section II of Internet Appendix D provides a thorough analysis of this issue. We find that a VaR constraint has a significant effect on the optimal portfolios of investors with low risk aversion, which are those with the largest exposure to risky assets in the unconstrained setting. The VaR-constrained investor tends to reduce the size of the myopic demand. Furthermore, because changes in covariances have a first-order impact on the VaR of the portfolio, the investor can even increase the covariance hedging demand, to exploit the spanning properties of the risky assets. Thus, in this setting, which is relevant for institutions subject to capital requirements or for asset managers with self-imposed risk management constraints, the impact of covariance risk can be economically very significant.

IV. Discussion and Conclusions

We develop a new multivariate framework for intertemporal portfolio choice, in which stochastic second moments of asset returns imply distinct motives for volatility and covariance hedging. The model is solved in closed form and is used to study volatility and covariance hedging in several realistic settings. We find that the multivariate nature of second moments has important consequences for optimal asset allocation: Hedging demands are typically four to five times larger than those of models with constant correlations or single-factor stochastic volatility. They include a substantial correlation hedging component, which tends to increase with the persistence of variance–covariance shocks, the strength of leverage effects, and the dimension of the investment opportunity set. These findings also arise within discrete-time versions of our model with short selling or VaR constraints.

The hedging demands due to variance–covariance risk are typically smaller than those found in the literature on intertemporal hedging with returns predictability. This feature follows mainly from the fact that returns span shocks in their covariance matrix much less than they typically do for shocks to the predictive variables. We do not explicitly incorporate Bayesian learning about model parameters. In continuous time, it is hard to motivate learning behavior about second moments of returns, because they are typically observable from the quadratic variation of returns. In discrete time, Bayesian learning about second moments can be more naturally considered. However, it is difficult to obtain tractable solutions for portfolio choice without a simple structure
for the variance–covariance process. Barberis (2000), among others, studies estimation risk about the parameters of a predictive equation in a model with homoskedastic returns, and finds that parameter uncertainty dramatically reduces the exposure to stocks over longer horizons. Our model is very different from that of Barberis. However, one might try to conclude by analogy that learning could also substantially reduce hedging demands in our case. Interesting evidence on this issue is given in Brandt et al. (2005). They develop a dynamic programming algorithm to efficiently solve the portfolio problem with predictability. When learning is considered, they find hedging demands that are comparable to those found in our paper. When learning is neglected, these policies are much higher than ours. Interestingly, the hedging demand reduction is almost entirely due to learning about the predictability equation: Learning about the variance–covariance matrix has a small influence on optimal portfolios. An interesting direction for future research could use the discrete-time Wishart process to study more systematically the portfolio implications of learning about the covariance matrix of returns.

REFERENCES


